Twisted duality and the Penrose polynomial

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Topics

- Twisted duality, a group action on ribbon graphs
- Relation to the medial graph
- A graph skein recursion polynomial
- Some properties of the Penrose polynomial

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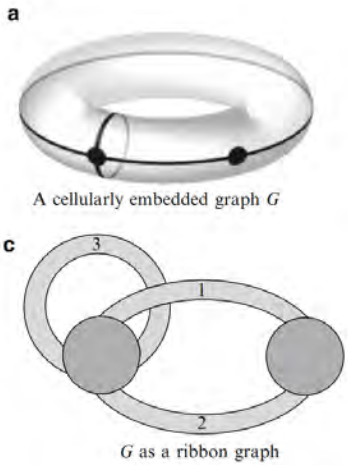
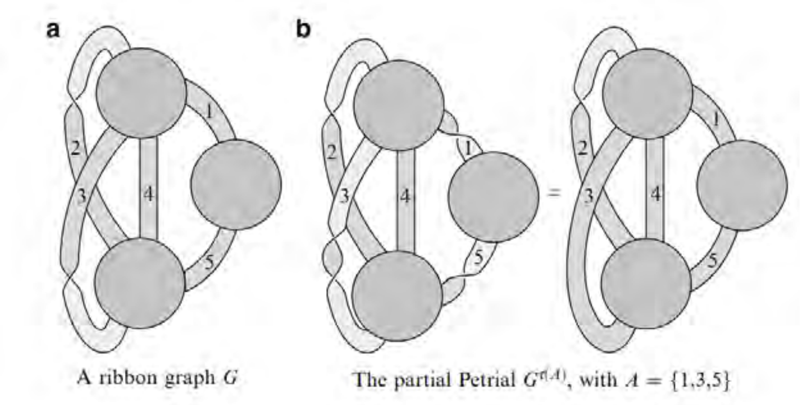
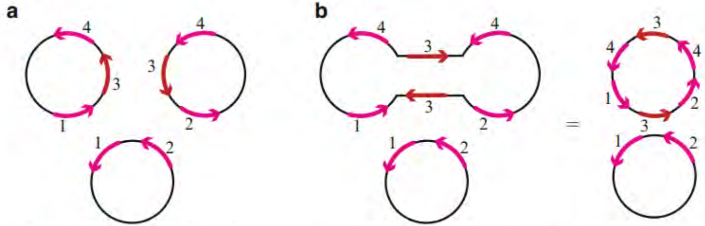
Embedded graphs



Partial petrials



Partial duals



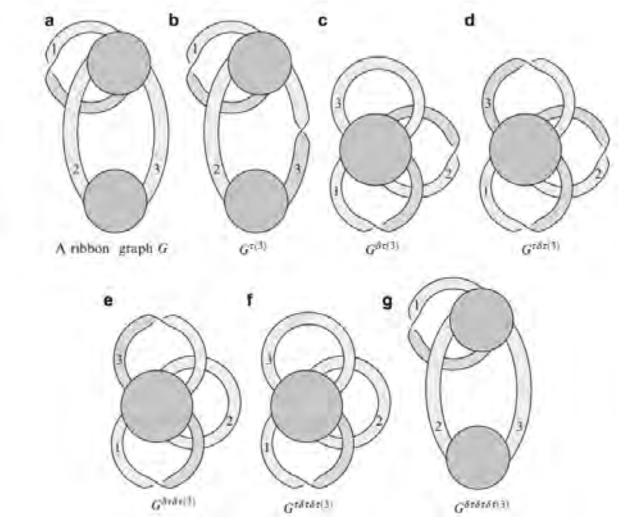
An arrow presentation for G

An arrow presentation for the partial dual $G^{\delta(3)}$

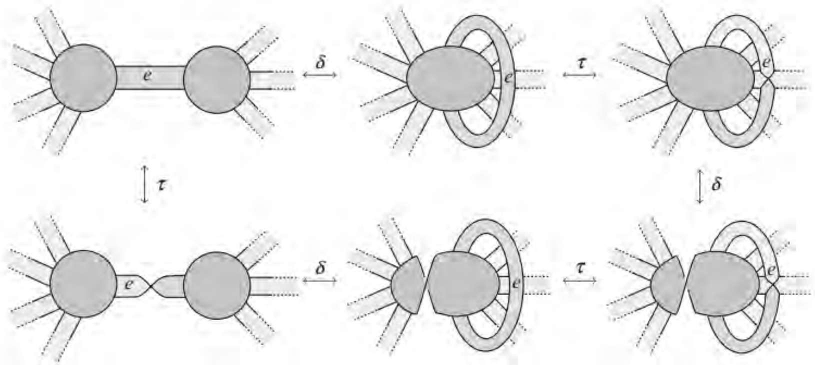
Properties of partial duals and petrials

- τ and δ act independently on edges and when applied twice return the original graph
- Let e, f be edges and A, B be edge sets, then:
- $G^{\tau^2(e)} = G^{\delta^2(e)} = G$
- $(G^{\tau(e)})^{\tau(f)} = (G^{\tau(f)})^{\tau(e)}, \quad (G^{\delta(e)})^{\delta(f)} = (G^{\delta(f)})^{\delta(e)}$
- $(G^{\tau(A)})^{\tau(B)} = (G^{\tau(B)})^{\tau(A)} = G^{\tau(A \Delta B)}$
- $(G^{\delta(A)})^{\delta(B)} = (G^{\delta(B)})^{\delta(A)} = G^{\delta(A \Delta B)}$
- $G^{\tau(E(G))} = G^{\times}, \quad G^{\delta(E(G))} = G^{*}$

Combining partial petrials and duals



Twisted duality



Twisted duality

- Consider τ and δ as operators on G on a single edge e
- We have: $\tau^2 = \delta^2 = (\delta\tau)^3 = 1$
- A group action $\mathfrak{G}$ on G with 6 distinct elements: $1, \tau, \delta, \delta\tau, \tau\delta, \tau\delta\tau$
- $G^{fg(e)} := (G^{g(e)})^{f(e)}$, where $f, g \in \mathfrak{G}$
- $\mathfrak{G}$ is isomorphic to S_3

The symmetric group S_3

- Permutations (self bijections) of a 3 element set
- E.g. $f : \{1,2,3\} \to \{1,2,3\} : f(1) = 2, f(2) = 3, f(3) = 1$
- In cycle notation: $f(x) = (1,2,3)$
- Elements of S_3: $\mathrm{id}, (1,2), (1,3), (2,3), (1,2,3), (3,2,1)$
- Nonabelian group under function composition,
  e.g. $(1,3) \circ (1,2) = (1,2,3), (1,2) \circ (1,3) = (3,2,1)$
- An isomorphism $\mathfrak{G} \to S_3$:
  $\varphi(\delta) = (1,2), \varphi(\tau) = (1,3), \varphi(\xi_1\xi_2) = \varphi(\xi_1) \circ \varphi(\xi_2)$

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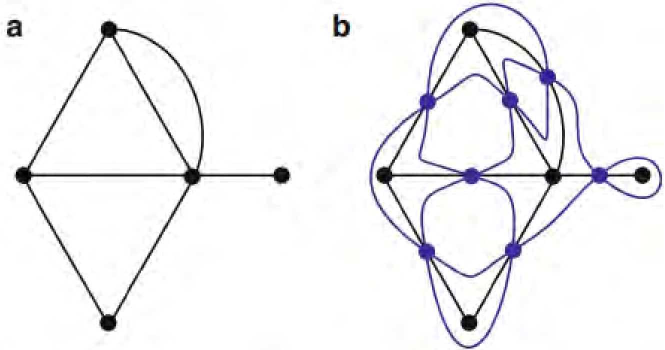
Medial graph example

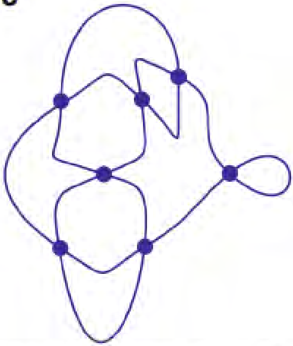


A cellularly embedded graph $G \subset S^2$

Forming the medial graph

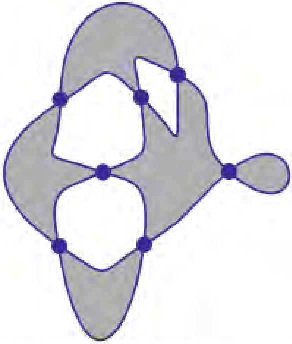
Medial graph example

c



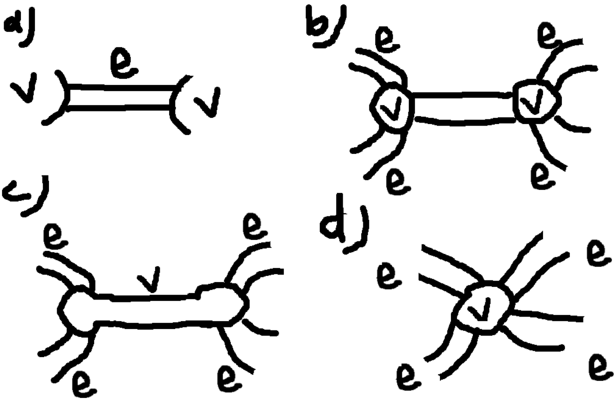
The medial graph $G_m \subset S^2$

d



The canonically checker-board coloured medial graph $G_m \subset S^2$

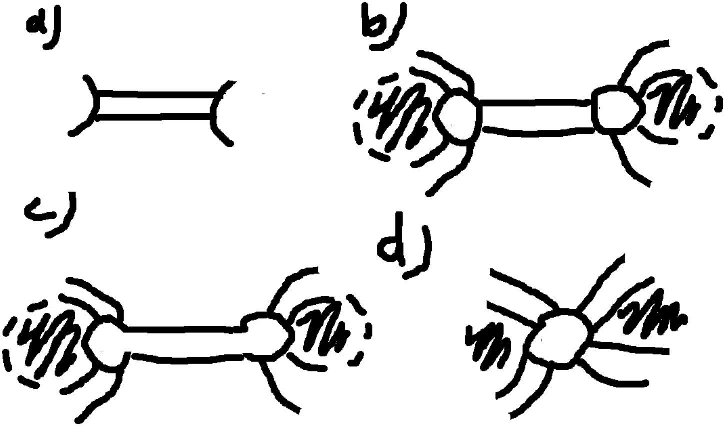
Medial graphs via the blow-up



a) A graph G b) The blow-up graph $G^\flat$ c-d) The medial graph G_m

Medial graphs via the blow-up

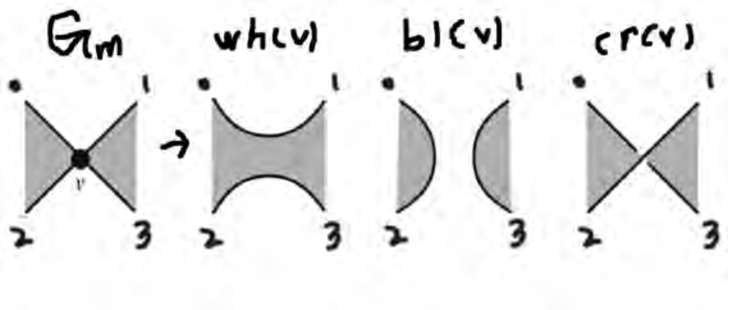
- F is medial iff F is 4-regular and checkerboard colorable



More properties of the medial graph

- d) $\to$ a) in the previous figure recovers G from G_m (and G^* if we switch colors)
- Uniqueness: G_m has only two checkerboard colorings, if $H_m = G_m = (G^*)_m$ then H induces a checkerboard coloring on F, hence $H = G$ or $H = G^*$
- We have, $H_m = G_m \iff H = G$ or $H = G^*$

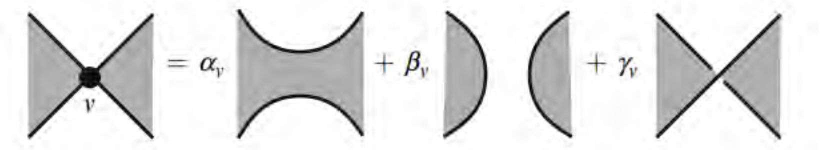
Vertex states of G_m



A medial graph and the white smoothing, black smoothing, and crossing vertex states of v

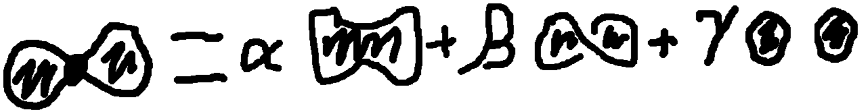
The topological transition polynomial

- $(\alpha_v, \beta_v, \gamma_v)$ assigned to each vertex for 3 possible states
- Vertex-less graphs are evaluated to t^c
  - $c =$ number of components
- $\alpha = 1, \beta = 0, \gamma = -1$ is the Penrose polynomial



The linear recursion relation of the topological transition polynomial

Example



$$= \gamma t^2 + (\alpha + \beta) t$$

The topological transition polynomial

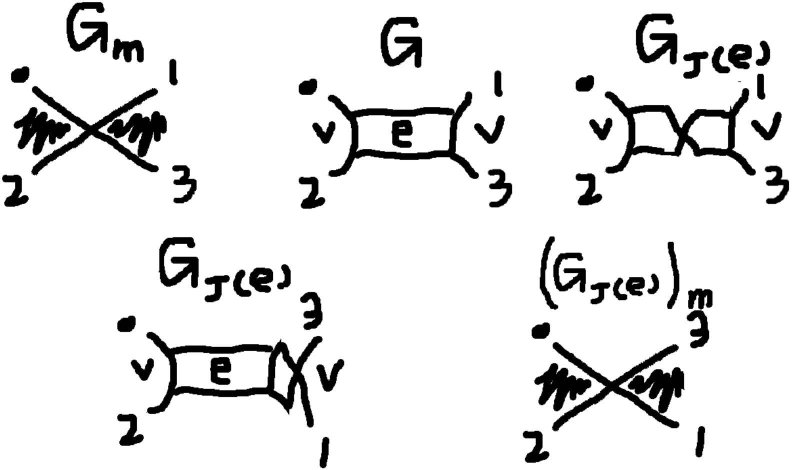
- For $s = (X, Y, Z)$ a partition of $V(G_m)$,
  $\omega(s) := (\prod_{v \in X} \alpha_v)(\prod_{v \in Y} \beta_v)(\prod_{v \in Z} \gamma_v)$
- $Q(G, (\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma}), t) := \sum_s \omega(s) t^{c(s)}$
- Note G in the definition, but $c(s)$ is the components of G_m
- Related to twisted duality:

$$Q(G, (\ldots, \alpha_v, \beta_v, \gamma_v, \ldots), t) = Q(G^{\delta(e)}, (\ldots, \beta_v, \alpha_v, \gamma_v, \ldots), t)$$

$$Q(G, (\ldots, \alpha_v, \beta_v, \gamma_v, \ldots), t) = Q(G^{\tau(e)}, (\ldots, \gamma_v, \beta_v, \alpha_v, \ldots), t)$$

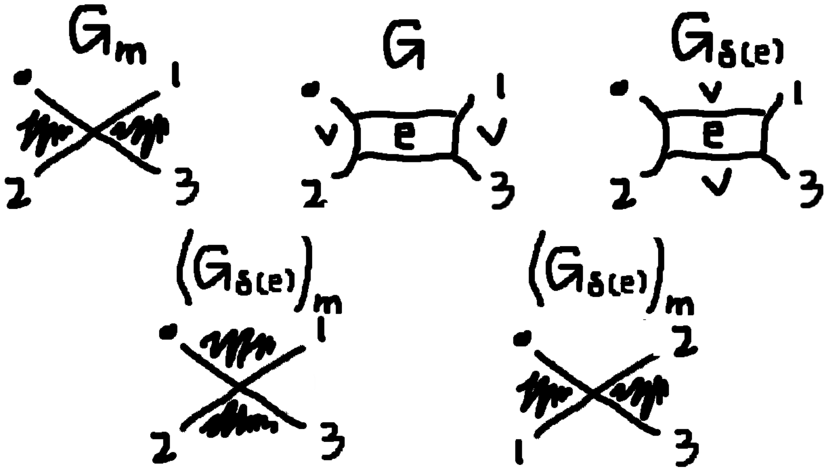
τ-permutation of vertex states

- White smoothing state of $(G^{\tau(e)})_m$ = crossing state of G_m (and vice versa)
- Hence the contributions to Q are the same



δ-permutation of vertex states

- White smoothing state of $(G^{\delta(e)})_m$ = black smoothing state of G_m (and vice versa)
- Again the contributions to Q are the same



Interaction of Q with twisted duality

$$\varphi : \mathfrak{G} \to S_3,\ \varphi(\delta) = (1,2), \varphi(\tau) = (1,3)$$

$$\forall \xi \in \mathfrak{G} : Q(G, (\ldots, \alpha_v, \beta_v, \gamma_v, \ldots), t) = Q(G^{\xi(e)}, (\ldots, \varphi(\xi)(\alpha_v, \beta_v, \gamma_v), \ldots), t)$$

$$\forall \boldsymbol{\xi} \in \mathfrak{G}^{V(G)} : Q(G, (\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma}), t) = Q(G^{\boldsymbol{\xi}(E(G))}, (\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma})^{\boldsymbol{\xi}(E(G))}), t)$$

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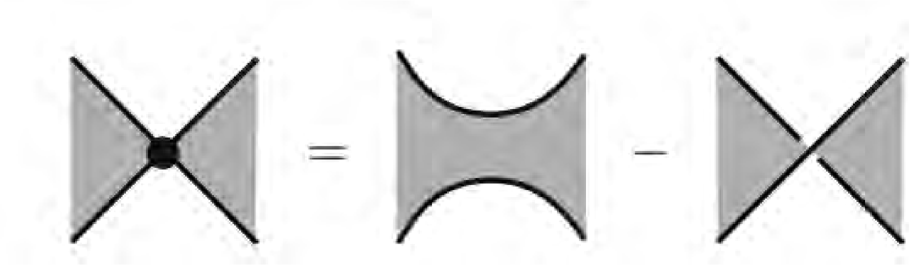
The Penrose polynomial

- Counts edge colorings of 4-regular plane graphs
- $P(G,t) = Q(G,(1,0,-1),t)$
- Four colour theorem $\iff$ $P(G,3) > 0$ $\forall$connected, bridgeless plane graphs G



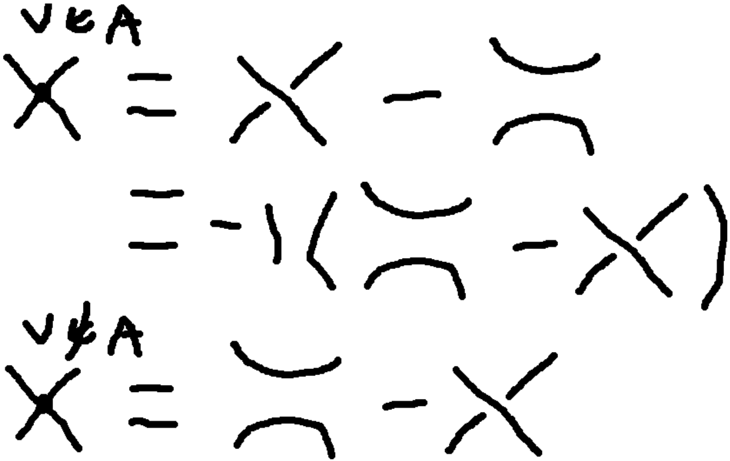
Interaction with τ

Proposition 4.19. *Let G be an embedded graph and $A \subseteq E(G)$. Then $P(G;\lambda) = (-1)^{|A|}P(G^{\tau(A)};\lambda)$ and in particular $|P(G;\lambda)|$ is invariant under partial Petriality.*

Proof. We have that

$$\begin{aligned} P(G^{\tau(A)};\lambda) &= Q(G^{\tau(A)};(\mathbf{1},\mathbf{0},-\mathbf{1}),\lambda) \\ &= Q((G^{\tau(A)})^{\tau(A)};(\mathbf{1},\mathbf{0},-\mathbf{1})^{\tau(A)},\lambda) \\ &= Q(G;(\mathbf{1},\mathbf{0},-\mathbf{1})^{\tau(A)},\lambda), \end{aligned}$$

Interaction with τ



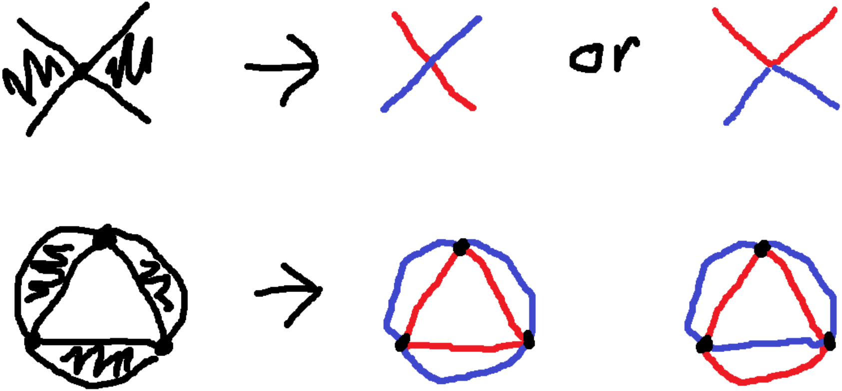
Interaction with τ

Corollary 4.20. *If G is a self-Petrial embedded graph, then $P(G;\lambda) = 0$ or $E(G)$ is even.*

Proof. G is self-Petrial if and only if $G^{\tau(E(G))} = G$. By Proposition 4.19, we then have $P(G;\lambda) = (-1)^{|E(G)|} P(G^{\tau(E(G))};\lambda)$, and the result follows. $\square$

Corollary 4.21. *If G is a self-Petrial plane graph with an odd number of edges, then G contains a bridge.*

n-valuations



n-valuations

Theorem 4.24. *If G is an embedded graph and $n \in \mathbb{N}$, then $P(G;n) = \sum(-1)^{cr(s)}$, where the sum is over all admissible n-valuations s of G_m and where $cr(s)$ is the number of crossing states in s.*

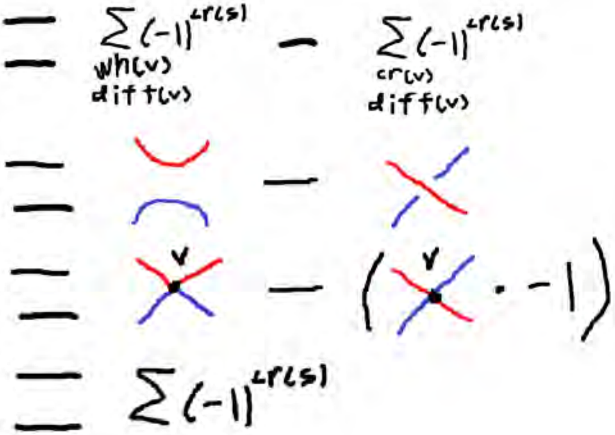
- Prove by induction on $V(G_m)$
- Base case: component diagram S, $n^{c(S)} = \sum 1 = \sum(-1)^{cr(S)}$

n-valuations

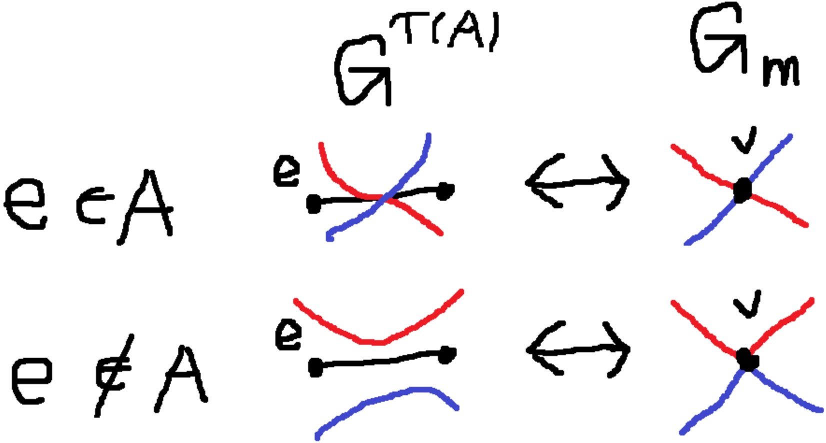
$$= \sum_{wh(v)} (-1)^{cr(s)} - \sum_{cr(v)} (-1)^{cr(s)}$$

$$= \sum_{\substack{wh(v)\\ same(v)}} (-1)^{cr(s)} + \sum_{\substack{wh(v)\\ diff(v)}} (-1)^{cr(s)} - \left(\sum_{\substack{cr(v)\\ same(v)}} (-1)^{cr(s)} + \sum_{\substack{cr(v)\\ diff(v)}} (-1)^{cr(s)} \right)$$

n-valuations



- A bijection between boundary components of $G^{\tau(A)}$ and colors in an n-valuation



n-valuations

- Boundary components of $G^{\tau(A)}$ are vertices of $(G^{\tau(A)})^*$
- Crossing/kissing components $G^{\tau(A)}$ are connected vertices of $(G^{\tau(A)})^*$
- Hence a proper n-vertex coloring of $(G^{\tau(A)})^*$ induces an n-valuation of G_m
- $cr(s)$ = twists of G
- We have:
  $P(G,n) = \sum_{\text{n-val}} (-1)^{cr(s)} = \sum_{A \subset E(G)} (-1)^{|A|} \chi((G^{\tau(A)})^*, n)$
- Corollary: $\chi(G^*, n) \leq P(G, n)$

Further reading

J. A. Ellis-Monaghan, I. Moffatt, *Graphs on Surfaces: Dualities, Polynomials, and Knots*, Springer, 2013.

Chapters 1, 2, and 4.3-4.4